FTUV/98-4 IFIC/98-4
January, 1998

An introduction to some novel applications of Lie algebra cohomology in mathematics and physics*

J. A. de Azcárraga†, J. M. Izquierdo‡ and J. C. Pérez Bueno†

† *Departamento de Física Teórica, Univ. de Valencia and IFIC, Centro Mixto Univ. de Valencia-CSIC, E–46100 Burjassot (Valencia), Spain.*

‡ *Departamento de Física Teórica, Universidad de Valladolid E–47011, Valladolid, Spain*

Abstract

After a self-contained introduction to Lie algebra cohomology, we present some recent applications in mathematics and in physics.

Contents

1 Preliminaries: $L_X,\ i_X,\ d$

Let us briefly recall here some basic definitions and formulae which will be useful later. Consider a uniparametric group of diffeomorphisms of a manifold M, $e^X : M \to M$, which takes a point $x \in M$ of local coordinates $\{x^i\}$ to $x'^i \simeq x^i + \epsilon^i(x)$ $(= x^i + X^i(x))$. Scalars and (covariant, say) tensors t_q $(q = 0, 1, 2, \ldots)$ transform as follows

$$\phi'(x') = \phi(x) \quad , \quad {t'}_i(x') = t_j(x)\frac{\partial x^j}{\partial x'^i} \quad , \quad {t'}_{i_1 i_2}(x') = t_{j_1 j_2}(x)\frac{\partial x^{j_1}}{\partial x'^{i_1}}\frac{\partial x^{j_2}}{\partial x'^{i_2}} \quad \ldots \quad . \tag{1.1}$$

In physics it is customary to define 'local' variations, which compare the transformed and original tensors at the same point x:

$$\delta\phi(x) \equiv \phi'(x) - \phi(x) \quad , \quad \delta t_i(x) \equiv {t'}_i(x) - t_i(x) \quad , \quad \ldots \quad . \tag{1.2}$$

Then, the first order variation defines the Lie derivative:

$$\begin{gathered} \delta_\epsilon \psi = -\epsilon^j(x)\partial_j\psi(x) := -L_X\psi \quad , \quad (\delta_\epsilon t)_i = -(\epsilon^j\partial_j t_i + (\partial_i\epsilon^j)t_j) := -(L_X t)_i \quad , \\ (\delta_\epsilon t)_{i_1 i_2} = -(\epsilon^j\partial_j t_{i_1 i_2} + (\partial_{i_1}\epsilon^j)t_{j i_2} + (\partial_{i_2}\epsilon^j)t_{i_1 j}) := -(L_X t)_{i_1 i_2} \quad . \end{gathered} \tag{1.3}$$

Eqs. (1.3) motivate the following general definition:

*To appear in the Proceedings of the VI Fall Workshop on *Geometry and physics* (Salamanca, September 1997).

Definition 1.1 (*Lie derivative*)
Let α be a (covariant, say) q tensor on M, $\alpha(x) = \alpha_{i_1 \dots i_q} dx^{i_1} \otimes \dots \otimes dx^{i_q}$, and $X = X^k \frac{\partial}{\partial x^k}$ a vector field $X \in \mathfrak{X}(M)$. The *Lie derivative* L_X of α with respect to X is locally given by

$$(L_X \alpha)_{i_1 \dots i_q} = X^k \frac{\partial \alpha_{i_1 \dots i_q}}{\partial x^k} + \alpha_{k i_2 \dots i_q} \frac{\partial X^k}{\partial x^{i_1}} + \dots + \alpha_{i_1 \dots i_{q-1} k} \frac{\partial X^k}{\partial x^{i_q}} \quad . \tag{1.4}$$

On a q-form $\alpha(x) = \dfrac{1}{q!} \alpha_{i_1 \dots i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q}$, $\alpha \in \wedge_q(M)$, $L_Y \alpha$ is defined by

$$(L_Y \alpha)(X_{i_1}, \dots, X_{i_q}) := Y \cdot \alpha(X_{i_1}, \dots, X_{i_q}) - \sum_{i=1}^{q} \alpha(X_{i_1}, \dots, [Y, X_i], \dots, X_{i_q}) \quad ; \tag{1.5}$$

on vector fields, $L_X Y = [X, Y]$. The action of L_X on tensors of any type t^p_q may be found using that L_X is a derivation,

$$L_X(t \otimes t') = (L_X t) \otimes t' + t \otimes L_X t' \quad . \tag{1.6}$$

Definition 1.2 (*Exterior derivative*)
The *exterior derivative* d is a derivation of degree $+1$, $d : \wedge_q(M) \to \wedge_{q+1}(M)$; it satisfies Leibniz's rule,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^q \alpha \wedge d\beta \quad , \quad \alpha \in \Lambda_q \quad , \tag{1.7}$$

and is nilpotent, $d^2 = 0$. On the q-form above, it is locally defined by

$$d\alpha = \frac{1}{q!} \frac{\partial \alpha_{i_1 \dots i_q}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_q} \quad . \tag{1.8}$$

The coordinate-free expression for the action of d is (Palais formula)

$$\begin{aligned} (d\alpha)(X_1, \dots, X_q, X_{q+1}) &:= \sum_{i=1}^{q+1} (-1)^{i+1} X_i \cdot \alpha(X_1, \dots, \hat{X}_i, \dots, X_{q+1}) \\ &\quad + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{q+1}) \quad . \end{aligned} \tag{1.9}$$

In particular, when α is a one-form,

$$d\alpha(X_1, X_2) = X_1 \cdot \alpha(X_2) - X_2 \cdot \alpha(X_1) - \alpha([X_1, X_2]) \quad . \tag{1.10}$$

Definition 1.3 (*Inner product*)
The *inner product* i_X is the derivation of degree -1 defined by

$$(i_X \alpha)(X_1, \dots, X_{q-1}) = \alpha(X, X_1, \dots, X_{q-1}) \quad . \tag{1.11}$$

On forms (Cartan decomposition of L_X),

$$L_X = i_X d + d i_X \quad , \tag{1.12}$$

from which $[L_X, d] = 0$ follows trivially. Other useful identity is

$$[L_X, i_Y] = i_{[X,Y]} \quad ; \tag{1.13}$$

from (1.12) and (1.13) it is easy to deduce that $[L_X, L_Y] = L_{[X,Y]}$.

2 Elementary differential geometry on Lie groups

Let G be a Lie group and let $L_{g'}g = g'g = R_g g'$ ($g',\ g \in G$) be the left and right actions $G \times G \to G$ with obvious notation. The left (right) invariant vector fields LIVF (RIVF) on G reproduce the commutator of the Lie algebra $\mathcal{G}$ of G

$$[X^L_{(i)}(g), X^L_{(j)}(g)] = C^k_{ij} X^L_{(k)}(g) \quad , \quad [X^R_{(i)}(g), X^R_{(j)}(g)] = -C^k_{ij} X^R_{(k)}(g) \quad , \quad C^\rho_{[i_1 i_2} C^\sigma_{\rho i_3]} = 0 \quad , \tag{2.1}$$

where the square bracket [ ] in the Jacobi identity (JI) means antisymmetrization of the indices i_1, i_2, i_3. In terms of the Lie derivative, the L- (R-) invariance conditions read[1]

$$L_{X^R_{(j)}(g)} X^L_{(i)}(g) = [X^R_{(j)}(g), X^L_{(i)}(g)] = 0 \quad , \quad L_{X^L_{(i)}(g)} X^R_{(j)}(g) = [X^L_{(i)}(g), X^R_{(j)}(g)] = 0 \quad . \tag{2.2}$$

Let $\omega^{L(i)}(g) \in \wedge_1(G)$ be the basis of LI one-forms dual to a basis of $\mathcal{G}$ given by LIVF ($\omega^{L(i)}(g)(X_{L(j)}(g)) = \delta^i_j$). Using (1.10), we get the Maurer-Cartan (MC) equations

$$d\omega^{L\,(i)}(g) = -\frac{1}{2} C^i_{jk} \omega^{L\,(j)}(g) \wedge \omega^{L\,(k)}(g) \quad . \tag{2.3}$$

In the language of forms, the JI in (2.1) follows from $d^2 = 0$. If the q-form α is LI

$$d\alpha^L(X^L_{i_1}, \ldots, X^L_{i_{q+1}}) = \sum_{s<t} (-1)^{s+t} \alpha^L([X^L_{i_s}, X^L_{i_t}], X^L_{i_1}, \ldots, \hat{X}^L_{i_s}, \ldots, \hat{X}^L_{i_t}, \ldots, X^L_{i_{q+1}}) \quad , \tag{2.4}$$

since $\alpha^L(X^L_1, \ldots, \hat{X}^L_i, \ldots, X^L_{q+1})$ in (1.9) is constant and does not contribute[2]. To facilitate the comparison with the generalized $\widetilde{d}_m$ to be introduced in Sec. 5, we note here that, with $\widetilde{d}_2 \equiv -d$, eq. (2.4) is equivalent to

$$\widetilde{d}_2 \alpha^L(X^L_{i_1}, \ldots, X^L_{i_{q+1}}) = \frac{1}{(2\cdot 2-2)!} \frac{1}{(q-1)!} \varepsilon^{j_1 \ldots j_{q+1}}_{i_1 \ldots i_{q+1}} \alpha^L([X^L_{j_1}, X^L_{j_2}], X^L_{j_3}, \ldots, X^L_{i_{q+1}}) \quad . \tag{2.5}$$

The MC equations may be written in a more compact way by introducing the (canonical) $\mathcal{G}$-valued LI one-form θ on G, $\theta(g) = \omega^{(i)}(g) X_{(i)}(g)$; then, MC equations read

$$d\theta = -\theta \wedge \theta = -\frac{1}{2}[\theta, \theta] \tag{2.6}$$

since, for $\mathcal{G}$-valued forms, $[\alpha, \beta] := \alpha^{(i)} \wedge \beta^{(j)} \otimes [X_{(i)}, X_{(j)}]$.

The transformation properties of $\omega^{(i)}(g)$ follow from (1.5):

$$L_{X_{(i)}(g)} \omega^{(j)}(g) = -C^j_{ik} \omega^{(k)}(g) \quad . \tag{2.7}$$

For a general LI q-form $\alpha(g) = \frac{1}{q!} \alpha_{i_1 \ldots i_q} \omega^{(i_1)}(g) \wedge \ldots \wedge \omega^{(i_q)}(g)$ on G

$$L_{X_{(i)}(g)} \alpha(g) = -\sum_{s=1}^{q} \frac{1}{q!} C^{i_s}_{ik} \alpha_{i_1 \ldots i_q} \omega^{(i_1)}(g) \wedge \ldots \wedge \widehat{\omega^{(i_s)}(g)} \wedge \omega^{(k)}(g) \wedge \ldots \wedge \omega^{(i_q)}(g) \quad . \tag{2.8}$$

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[1] The superindex L (R) in the fields refers to the left (right) invariance of them; LIVF (RIVF) generate right (left) translations.

[2] From now on we shall assume that vector fields and forms are left invariant (*i.e.*, $X \in \mathfrak{X}^L(G)$, etc.) and drop the superindex L. Superindices $L,\ R$ will be used to avoid confusion when both LI and RI vector fields appear.

3 Lie algebra cohomology: a brief introduction

3.1 Lie algebra cohomology

Definition 3.1 (*V-valued n-dimensional cochains on $\mathcal{G}$*)
Let $\mathcal{G}$ be a Lie algebra and V a vector space. A V-valued n-cochain Ω_n on $\mathcal{G}$ is a skew-symmetric n-linear mapping

$$\Omega_n : \mathcal{G} \wedge \overset{n}{\cdots} \wedge \mathcal{G} \to V \quad , \quad \Omega_n^A = \frac{1}{n!}\Omega^A_{i_1 \ldots i_n}\omega^{i_1} \wedge \ldots \wedge \omega^{i_n} \quad , \tag{3.1}$$

where $\{\omega^{(i)}\}$ is a basis of $\mathcal{G}^*$ and the superindex A labels the components in V. The (abelian) group of all n-cochains is denoted by $C^n(\mathcal{G}, V)$.

Definition 3.2 (*Coboundary operator* (for the left action ρ of $\mathcal{G}$ on V))
Let V be a left $\rho(\mathcal{G})$-module, where ρ is a representation of the Lie algebra $\mathcal{G}$, $\rho(X_i)^A_{.C}\rho(X_j)^C_{.B} - \rho(X_j)^A_{.C}\rho(X_i)^C_{.B} = \rho([X_i, X_j])^A_{.B}$. The coboundary operator $s : C^n(\mathcal{G}, V) \to C^{n+1}(\mathcal{G}, V)$ is defined by

$$\begin{aligned}(s\Omega_n)^A(X_1, ..., X_{n+1}) := &\sum_{i=1}^{n+1}(-)^{i+1}\rho(X_i)^A_{.B}\left(\Omega^B_n(X_1, ..., \hat{X}_i, ..., X_{n+1})\right) \\ &+ \sum_{\substack{j,k=1 \\ j<k}}^{n+1}(-)^{j+k}\Omega^A_n([X_j, X_k],\ X_1, ..., \hat{X}_j, ..., \hat{X}_k, ..., X_{n+1}) \quad .\end{aligned} \tag{3.2}$$

Proposition 3.1
The *Lie algebra cohomology* operator s is nilpotent, $s^2 = 0$.

Proof. Looking at (1.9), s in (3.2) may be at this stage formally written as

$$(s)^A_{.B} = \delta^A_B d + \rho(X_i)^A_{.B}\omega^i \quad , \quad (s = d + \rho(X_i)\omega^i) \quad . \tag{3.3}$$

Then, the proposition follows from the fact that

$$\begin{aligned}s^2 = &\quad (\rho(X_i)\omega^i + d)(\rho(X_j)\omega^j + d) = \rho(X_i)\rho(X_j)\omega^i \wedge \omega^j + \rho(X_i)\omega^i d + \rho(X_j)d\omega^j + d^2 \\ = &\quad -\frac{1}{2}\rho(X_j)C^j_{lk}\omega^l \wedge \omega^k + \frac{1}{2}[\rho(X_i), \rho(X_j)]\omega^i \wedge \omega^j = 0 \quad .\end{aligned} \tag{3.4}$$

Definition 3.3 (*n-th cohomology group*)
An n-cochain Ω_n is a cocycle, $\Omega_n \in Z^n_\rho(\mathcal{G}, V)$, when $s\Omega_n = 0$. If a cocycle Ω_n may be written as $\Omega_n = s\Omega'_{n-1}$ in terms of an $(n-1)$-cochain Ω'_{n-1}, Ω_n is a coboundary, $\Omega_n \in B^n_\rho(\mathcal{G}, V)$. The n-th Lie algebra cohomology group $H^n_\rho(\mathcal{G}, V)$ is defined by

$$H^n_\rho(\mathcal{G}, V) = Z^n_\rho(\mathcal{G}, V)/B^n_\rho(\mathcal{G}, V) \quad . \tag{3.5}$$

3.2 Chevalley-Eilenberg formulation

Let V be $\mathbb{R}$, ρ trivial. Then the first term in (3.2) is not present and, on LI one-forms, s and d act in the same manner. Since there is a one-to-one correspondence between n-antisymmetric maps on $\mathcal{G}$ and LI n-forms on G, an n-cochain in $C^n(\mathcal{G},\mathbb{R})$ may also be given by the LI form on G

$$\Omega(g) = \frac{1}{n!}\Omega_{i_1 \ldots i_n}\omega^{(i_1)}(g) \wedge \ldots \wedge \omega^{(i_n)}(g) \tag{3.6}$$

and the Lie algebra cohomology coboundary operator is now d [1] (the explicit dependence of the forms $\Omega(g)$, $\omega^i(g)$ on g will be omitted henceforth).

Remark. It should be noticed that the Lie algebra (CE) cohomology is in general different from the de Rham cohomology: a form β on G may be de Rham exact, $\beta = d\alpha$, but the potential form α might not be a cochain *i.e.*, a LI form[3]. Nevertheless, for G compact (see Proposition 4.7) $H_{DR}(G) = H_0(\mathcal{G},\mathbb{R})$.

Example 3.1
Let $\mathcal{G}$ be the abelian two-dimensional algebra. The corresponding Lie group is $\mathbb{R}^2$, which is de Rham trivial. However, the translation algebra $\mathbb{R}^2$ has non-trivial Lie algebra cohomology, and in fact it admits a non-trivial two-cocycle giving rise to the three-dimensional Heisenberg-Weyl algebra.

3.3 Whitehead's lemma for vector valued cohomology

Lemma 3.1 (*Whitehead's lemma*)
Let $\mathcal{G}$ be a finite-dimensional semisimple Lie algebra over a field of characteristic zero and let V be a finite-dimensional irreducible $\rho(\mathcal{G})$-module such that $\rho(\mathcal{G})V \neq 0$ (ρ *non-trivial*). Then,

$$H^q_\rho(\mathcal{G}, V) = 0 \quad \forall\, q \geq 0 \quad . \tag{3.7}$$

If $q = 0$, the non-triviality of ρ and the irreducibility imply that $\rho(\mathcal{G}) \cdot v = 0$ ($v \in V$) holds only for $v = 0$.

Proof. Since $\mathcal{G}$ is semi-simple, the Cartan-Killing metric g_{ij} is invertible, $g^{ij}g_{jk} = \delta^i_k$. Let τ be the operator on the space of q-cochains $\tau : C^q(\mathcal{G}, V) \to C^{q-1}(\mathcal{G}, V)$ defined by

$$(\tau\Omega)^A_{i_1 \ldots i_{q-1}} = g^{ij}\rho(X_i)^A_{.B}\Omega^B_{j i_1 \ldots i_{q-1}} \quad . \tag{3.8}$$

It is not difficult to check that on cochains the Laplacian-like operator $(s\tau + \tau s)$ gives[4]

$$[(s\tau + \tau s)\Omega]^A_{i_1 \ldots i_q} = \Omega^B_{i_1 \ldots i_q} I_2(\rho)^A_{.B} \quad , \tag{3.10}$$

[3] This is, *e.g.*, the case for certain forms which appear in the theory of supersymmetric extended objects (superstrings). This is not surprising due to the absence of global considerations in the fermionic sector of supersymmetry. The Lie algebra cohomology notions are easily extended to the 'super Lie' case (see *e.g.*, [2] for references on these subjects).

[4] For instance, for a two-cochain eq. (3.10) reads

$$\begin{aligned}
[(s\tau + \tau s)\Omega]^A_{ij} = \; & g^{kl}\rho(X_i)^A_{.B}\rho(X_k)^B_{.C}\Omega^C_{lj} - g^{kl}\rho(X_j)^A_{.B}\rho(X_k)^B_{.C}\Omega^C_{li} - g^{kl}\rho(X_k)^A_{.B}C^m_{ij}\Omega^B_{lm} \\
+ \; & g^{kl}\rho(X_k)^A_{.B}\rho(X_l)^B_{.C}\Omega^C_{ij} + g^{kl}\rho(X_k)^A_{.B}\rho(X_i)^B_{.C}\Omega^C_{jl} + g^{kl}\rho(X_k)^A_{.B}\rho(X_j)^B_{.C}\Omega^C_{li} \\
- \; & g^{kl}\rho(X_k)^A_{.B}C^m_{ij}\Omega^B_{ml} - g^{kl}\rho(X_k)^A_{.B}C^m_{li}\Omega^B_{mj} - g^{kl}\rho(X_k)^A_{.B}C^m_{jl}\Omega^B_{mi} \\
= \; & g^{kl}[\rho(X_i),\rho(X_k)]^A_{.B}\Omega^B_{lj} - g^{kl}[\rho(X_j),\rho(X_k)]^A_{.B}\Omega^B_{li} + I_2(\rho)^A_{.B}\Omega^B_{ij} \\
- \; & g^{kl}\rho(X_k)^A_{.B}C^m_{li}\Omega^B_{mj} - g^{kl}\rho(X_k)^A_{.B}C^m_{jl}\Omega^B_{mi} = I_2(\rho)^A_{.B}\Omega^B_{ij} \quad .
\end{aligned} \tag{3.9}$$

where $I_2(\rho)^A_{\cdot B} = g^{ij}(\rho(X_i)\rho(X_j))^A_{\cdot B}$ is the quadratic Casimir operator in the representation ρ. By Schur's lemma it is proportional to the unit matrix. Hence, applying (3.10) to $\Omega \in Z^q_\rho(\mathcal{G}, V)$ we find

$$s\tau\Omega = \Omega I_2(\rho) \;\Rightarrow\; s(\tau\Omega I_2(\rho)^{-1}) = \Omega \quad . \tag{3.11}$$

Thus, Ω is the coboundary generated by the cochain $\tau\Omega I_2(\rho)^{-1} \in C^{q-1}_\rho(\mathcal{G}, V)$, *q.e.d.*

For semisimple algebras and $\rho = 0$ we also have $H^1_0 = 0$ and $H^2_0 = 0$, but already $H^3_0 \neq 0$.

3.4 Lie algebra cohomology *à la* BRST

In many physical applications it is convenient to introduce the so-called BRST operator (for Becchi, Rouet, Stora and Tyutin) acting on the space of BRST cochains. To this aim let us introduce anticommuting, 'odd' objects (in physics they correspond to the *ghosts*)

$$c^i c^j = -c^j c^i \quad , \quad i, j = 1, \ldots, \dim \mathcal{G} \quad . \tag{3.12}$$

The operator $\mathfrak{s}$ defined by

$$\mathfrak{s} := \frac{1}{2} C^k_{ij} c^j c^i \frac{\partial}{\partial c^k} \tag{3.13}$$

acts on the ghosts as the exterior derivative d acts on LI one-forms ($\mathfrak{s}c^k = -1/2 C^k_{ij} c^i c^j$, cf. (2.3)) and, as d, is nilpotent, $\mathfrak{s}^2 = 0$. For the cohomology associated with a non-trivial action ρ of $\mathcal{G}$ on V we introduce the BRST $\tilde{s}$ operator

$$\tilde{s} := c^i \rho(X_i) + \frac{1}{2} C^k_{ij} c^j c^i \frac{\partial}{\partial c^k} \quad . \tag{3.14}$$

Proposition 3.2
The BRST operator $\tilde{s}$ is nilpotent $\tilde{s}^2 = 0$.

Proof. First, we rewrite $\tilde{s}$ as

$$\tilde{s} = c^i N_{(i)} \quad , \quad N_{(i)} = \rho(X_i) + \frac{1}{2} C^k_{ji} c^j \frac{\partial}{\partial c^k} \equiv N^1_{(i)} + \frac{1}{2} N^2_{(i)} \quad . \tag{3.15}$$

The operator $N_{(i)}$ has two different pieces N^1 and N^2, each of them carrying a representation of $\mathcal{G}$ so that $[N_{(i)}, N_{(j)}] = C^k_{ij}(N^1_{(k)} + \frac{1}{4} N^2_{(k)})$. Thus,

$$\begin{aligned} \tilde{s}^2 &= c^i N_{(i)} c^j N_{(j)} = \frac{1}{2} c^i c^j [N_{(i)}, N_{(j)}] + c^i (N_{(i)}.c^j) N_{(j)} \\ &= \frac{1}{2} c^i c^j C^k_{ij} (N^1_{(k)} + \frac{1}{4} N^2_{(k)}) + \frac{1}{2} c^i c^j C^k_{ji} N_{(k)} = \frac{1}{2} c^i c^j C^k_{ij} N^1_{(k)} + \frac{1}{2} c^i c^j C^k_{ji} N^1_{(k)} = 0 \ , \end{aligned} \tag{3.16}$$

by virtue of the anticommutativity of the c's, and using that $c^i c^j C^k_{ij} N^2_{(k)} = 0$ and $N_{(i)}.c^j = \frac{1}{2} c^k C^j_{ki}$. Thus, on the 'BRST-cochains'

$$\tilde{\Omega}^A_n = \frac{1}{n!} \Omega^A_{i_1 \ldots i_n} c^{i_1} \ldots c^{i_n} \quad , \tag{3.17}$$

the action of $\tilde{s}$ is the same as that of s in (3.2) and may be used to define the Lie algebra cohomology.

4 Symmetric polynomials and higher order cocycles

4.1 Symmetric invariant tensors and higher order Casimirs

From now on, we shall restrict ourselves to simple Lie groups and algebras; by virtue of Lemma 3.1, only the $\rho = 0$ case is interesting. The non-trivial cohomology groups are related to the primitive symmetric invariant tensors [3, 4, 5, 6, 7, 8, 9, 10] on $\mathcal{G}$, which in turn determine Casimir elements in the universal enveloping algebra $\mathcal{U}(\mathcal{G})$.

Definition 4.1 (*Symmetric and invariant polynomials on* $\mathcal{G}$)
A symmetric polynomial on $\mathcal{G}$ is given by a symmetric covariant LI tensor. It may be expressed as a LI covariant tensor on G, $k = k_{i_1 \dots i_m} \omega^{i_1} \otimes \dots \otimes \omega^{i_m}$ with symmetric constant coordinates $k_{i_1 \dots i_m}$. k is said to be an invariant or (ad-invariant) symmetric polynomial if it is also right-invariant, *i.e.* if $L_{X_l} k = 0 \; \forall \, X_l \in \mathfrak{X}^L(G)$. Indeed, using (2.8), we find that

$$L_{X_l} k = 0 \;\Rightarrow\; C^s_{l i_1} k_{s i_2 \dots i_m} + C^s_{l i_2} k_{i_1 s \dots i_m} + \dots + C^s_{l i_m} k_{i_1 \dots i_{m-1} s} = 0 \quad . \tag{4.1}$$

Since the coordinates of k are given by $k_{i_1 \dots i_m} = k(X_{i_1}, \dots, X_{i_m})$, eq. (4.1) is equivalent to stating that k is ad-invariant, *i.e.*,

$$k([X_l, X_{i_1}], \dots, X_{i_m}) + k(X_{i_1}, [X_l, X_{i_2}], \dots, X_{i_m}) + \dots + k(X_{i_1}, \dots, [X_l, X_{i_m}]) = 0 \tag{4.2}$$

or, equivalently,

$$k(Ad\, g\, X_{i_1}, \dots, Ad\, g\, X_{i_m}) = k(X_{i_1}, \dots, X_{i_m}) \quad , \tag{4.3}$$

from which eq. (4.2) follows by taking the derivative $\partial / \partial g^l$ in $g = e$.

The invariant symmetric polynomials just described can be used to construct Casimir elements of the enveloping algebra $\mathcal{U}(\mathcal{G})$ of $\mathcal{G}$ in the following way

Proposition 4.1
Let k be a symmetric invariant tensor. Then $k^{i_1 \dots i_m} X_{i_1} \dots X_{i_m}$ (coordinate indices of k raised using the Killing metric), is a Casimir of order m, *i.e.* $[k^{i_1 \dots i_m} X_{i_1} \dots X_{i_m}, Y] = 0 \; \forall \, Y \in \mathcal{G}$.

Proof.

$$\begin{aligned} [k^{i_1 \dots i_m} X_{i_1} \dots X_{i_m}, X_s] &= \sum_{j=1}^{m} k^{i_1 \dots i_m} X_{i_1} \dots [X_{i_j}, X_s] \dots X_{i_m} \\ &= \sum_{j=1}^{m} k^{i_1 \dots i_m} X_{i_1} \dots C^t_{i_j s} X_t \dots X_{i_m} = 0 \end{aligned} \tag{4.4}$$

by (4.1), *q.e.d.*

A well-known way of obtaining symmetric (ad-)invariant polynomials (used *e.g.*, in the construction of characteristic classes) is given by

Proposition 4.2
Let X_i denote now a representation of $\mathcal{G}$. Then, the symmetrized trace

$$k_{i_1 \dots i_m} = \mathrm{sTr}(X_{i_1} \dots X_{i_m}) \tag{4.5}$$

defines a symmetric invariant polynomial.

Proof. k is symmetric by construction and the ad-invariance is obvious since $Adg\, X := gXg^{-1}$, *q.e.d.*

The simplest illustration of (4.5) is the Killing tensor for a simple Lie algebra $\mathcal{G}$, $k_{ij} = \mathrm{Tr}(ad\, X_i\, ad\, X_j)$; its associated Casimir is the second order Casimir I_2.

Example 4.1
Let $\mathcal{G} = su(n)$, $n \geq 2$, and let X_i be (hermitian) matrices in the defining representation. Then

$$\mathrm{sTr}(X_i X_j X_k) \propto 2\mathrm{Tr}(\{X_i, X_j\} X_k) = d_{ijk} \quad , \tag{4.6}$$

using that, for the $su(n)$ algebra, $\{X_i, X_j\} = c\delta_{ij} + d_{ijl} X_l$, $\mathrm{Tr}(X_k) = 0$ and $\mathrm{Tr}(X_i X_j) = \frac{1}{2}\delta_{ij}$. This third order polynomial leads to the Casimir I_3; for $su(2)$ only k_{ij} and I_2 exist.

Example 4.2
In the case $\mathcal{G} = su(n)$, $n \geq 4$, we have a fourth order polynomial

$$\mathrm{sTr}(X_{i_1} X_{i_2} X_{i_3} X_{i_4}) \propto d_{(i_1 i_2 l} d_{l i_3) i_4} + 2c\delta_{(i_1 i_2}\delta_{i_3) i_4} \quad , \tag{4.7}$$

where ( ) indicates symmetrization. The first term leads to a fourth order Casimir I_4 whereas the second one includes (see [11]) a term in I_2^2.

Eq. (4.7) deserves a comment. The first part $d_{(i_1 i_2 l} d_{l i_3) i_4}$ generalizes easily to higher n by nesting more d's, leading to the Klein [5] form of the $su(n)$ Casimirs. The second part includes a term that is the product of Casimirs of order two: it is not *primitive*.

Definition 4.2 (*Primitive symmetric invariant polynomials*)
A symmetric invariant polynomial $k_{i_1 \ldots i_m}$ on $\mathcal{G}$ is called primitive if it is not of the form

$$k_{i_1 \ldots i_m} = k^{(p)}_{(i_1 \ldots i_p} k^{(q)}_{i_{p+1} \ldots i_m)} \ , \quad p + q = m \quad , \tag{4.8}$$

where $k^{(p)}$ and $k^{(q)}$ are two lower order symmetric invariant polynomials.

Of course, we could also have considered eq. (4.7) for $su(3)$, but then it would not have led to a fourth-order primitive polynomial, since $su(3)$ is a rank 2 algebra. Indeed, $d_{(i_1 i_2 l} d_{l i_3) i_4}$ is not primitive for $su(3)$ and can be written in terms of $\delta_{i_1 i_2}$ as in (4.8) (see, *e.g.*, [12]; see also [11] and references therein). In general, for a simple algebra of rank l there are l invariant primitive polynomials and Casimirs [3, 4, 5, 6, 7, 8, 9, 10] and, as we shall show now, l primitive Lie algebra cohomology cocycles.

4.2 Cocycles from invariant polynomials

We make now explicit the connection between the invariant polynomials and the non-trivial cocycles of a simple Lie algebra $\mathcal{G}$. To do this we may use the particular case of $\mathcal{G} = su(n)$ as a guide. On the manifold of the group $SU(n)$ one can construct the *odd* q-form

$$\Omega = \frac{1}{q!}\mathrm{Tr}(\theta \wedge \stackrel{q}{\cdots} \wedge \theta) \quad , \tag{4.9}$$

where $\theta = \omega^i X_i$ and we take $\{X_i\}$ in the defining representation; q has to be odd since otherwise Ω would be zero (by virtue of the cyclic property of the trace and the anticommutativity of one-forms).

Proposition 4.3
The LI odd form Ω on G in (4.9) is a non-trivial (CE) Lie algebra cohomology cocycle.

Proof. Since Ω is LI by construction, it is sufficient to show that Ω is closed and that it is not the differential of another LI form (*i.e.* it is not a coboundary). By using (2.6) we get

$$d\Omega = -\frac{1}{(q-1)!}\mathrm{Tr}(\theta \wedge \overset{q+1}{\cdots} \wedge \theta) = 0 \quad , \tag{4.10}$$

since $q+1$ is even. Suppose now that $\Omega = d\Omega_{q-1}$, with Ω_{q-1} LI. Then Ω_{q-1} would be of the form (4.9) and hence zero because $q-1$ is also even, *q.e.d.*

All non-trivial q-cocycles in $H_0^q(su(n), \mathbb{R})$ are of the form (4.9). The fact that they are closed and non-exact ($SU(n)$ is compact) allows us to use them to construct Wess-Zumino-Witten [13, 14] terms on the group manifold (see also [15]).

Let us set $q = 2m-1$. The form Ω expressed in coordinates is

$$\begin{aligned}\Omega &= \frac{1}{q!}\mathrm{Tr}(X_{i_1} \ldots X_{i_{2m-1}})\omega^{i_1} \wedge \ldots \wedge \omega^{i_{2m-1}} \\ &\propto \mathrm{Tr}([X_{i_1}, X_{i_2}][X_{i_3}, X_{i_4}] \ldots [X_{i_{2m-3}}, X_{i_{2m-2}}]X_{i_{2m-1}})\omega^{i_1} \wedge \ldots \wedge \omega^{i_{2m-1}} \\ &= \mathrm{Tr}(X_{l_1} \ldots X_{l_{m-1}} X_\sigma) C^{l_1}_{i_1 i_2} \ldots C^{l_{m-1}}_{i_{2m-3} i_{2m-2}} \omega^{i_1} \wedge \ldots \wedge \omega^{i_{2m-2}} \wedge \omega^\sigma \quad .\end{aligned} \tag{4.11}$$

We see here how the order m symmetric (there is symmetry in $l_1 \ldots l_{m-1}$ because of the ω^i's) invariant polynomial $\mathrm{Tr}(X_{l-1} \ldots X_{l_{m-1}} X_\sigma)$ appears in this context. Conversely, the following statement holds

Proposition 4.4
Let $k_{i_1 \ldots i_m}$ be a symmetric invariant polynomial. Then, the polynomial

$$\Omega_{\rho i_2 \ldots i_{2m-2} \sigma} = C^{l_1}_{j_2 j_3} \ldots C^{l_{m-1}}_{j_{2m-2} \sigma} k_{\rho l_1 \ldots l_{m-1}} \varepsilon^{j_2 \ldots j_{2m-2}}_{i_2 \ldots i_{2m-2}} \tag{4.12}$$

is skew-symmetric and defines the closed form (cocycle)

$$\Omega = \frac{1}{(2m-1)!}\Omega_{\rho i_2 \ldots i_{2m-2} \sigma} \omega^\rho \wedge \omega^{i_2} \wedge \ldots \wedge \omega^{i_{2m-2}} \wedge \omega^\sigma \quad . \tag{4.13}$$

Proof. To check the complete skew-symmetry of $\Omega_{\rho i_2 \ldots i_{2m-2} \sigma}$ in (4.12), it is sufficient, due to the ε, to show the antisymmetry in ρ and σ. This is done by using the invariance of k (4.1) and the symmetry properties of k and ε to rewrite $\Omega_{\rho i_2 \ldots i_{2m-2} \sigma}$ as the sum of two terms. The first one,

$$\begin{aligned}&\sum_{s=1}^{m-2} \varepsilon^{j_2 \ldots j_{2s} j_{2s+1} j_{2m-2} j_{2s+2} \ldots j_{2m-3}}_{i_2 \ldots i_{2m-2}} k_{\rho l_1 \ldots l_{s-1} l_{m-1} l_s \ldots l_{m-2} \sigma} \\ &C^{l_1}_{j_2 j_3} \ldots C^{l_s}_{j_{2s} j_{2s+1}} C^{l_{m-1}}_{l_s j_{2m-2}} C^{l_{s+1}}_{j_{2s+2} j_{2s+3}} \ldots C^{l_{m-2}}_{j_{2m-4} j_{2m-3}}\end{aligned} \tag{4.14}$$

vanishes due to the Jacobi identity in (2.1), and the second one is

$$\Omega_{\rho i_2 \ldots i_{2m-2} \sigma} = -\varepsilon^{j_2 \ldots j_{2m-2}}_{i_2 \ldots i_{2m-2}} k_{\sigma l_1 \ldots l_{m-1}} C^{l_1}_{j_2 j_3} \ldots C^{l_{m-1}}_{j_{2m-2} \rho} = -\Omega_{\sigma i_2 \ldots i_{2m-2} \rho} \quad . \tag{4.15}$$

To show that $d\Omega = 0$ we make use of the fact that any bi-invariant form (*i.e.*, a form that is both LI and RI) is closed (see, *e.g.*, [2]). Since Ω is LI by construction, we only need to prove its right-invariance, but

$$\Omega \propto \mathrm{Tr}(\theta \wedge \overset{2m-1}{\cdots} \wedge \theta) \tag{4.16}$$

is obviously RI since $R_g^* \theta = Adg^{-1}\theta$, *q.e.d.*

Without discussing the origin of the invariant polynomials for the different groups [3, 4, 5, 6, 7, 8, 9, 10, 11], we may conclude that to each symmetric primitive invariant polynomial of order m we can associate a Lie algebra cohomology $(2m-1)$-cocycle (see [11] for practical details). The question that immediately arises is whether this construction may be extended since, from a set of l primitive invariant polynomials, we can obtain an arbitrary number of non-primitive polynomials (see eq. (4.8)). This question is answered negatively by Proposition 4.5 and Corollary 4.1 below.

Proposition 4.5
Let $k_{i_1\ldots i_m}$ be a symmetric G-invariant polynomial. Then,

$$\epsilon^{j_1\ldots j_{2m}}_{i_1\ldots i_{2m}} C^{l_1}_{j_1 j_2}\ldots C^{l_m}_{j_{2m-1}j_{2m}} k_{l_1\ldots l_m} = 0 \quad . \tag{4.17}$$

Proof. By replacing $C^{l_m}_{j_{2m-1}j_{2m}} k_{l_1\ldots lm}$ in the $l.h.s$ of (4.17) by the other terms in (4.1) we get

$$\epsilon^{j_1\ldots j_{2m}}_{i_1\ldots i_{2m}} C^{l_1}_{j_1 j_2}\ldots C^{l_{m-1}}_{j_{2m-3}j_{2m-2}} \Big(\sum_{s=1}^{m-1} C^{k}_{j_{2m-1}l_s} k_{l_1\ldots l_{s-1}kl_{s+1}\ldots l_{m-1}\, j_{2m}}\Big) \quad , \tag{4.18}$$

which is zero due to the JI, *q.e.d.*

Corollary 4.1
Let k be a non-primitive symmetric invariant polynomial (4.8), Then the $(2m-1)$-cocycle Ω associated to it (4.13) is zero.

Thus, to a *primitive* symmetric m-polynomial it is possible to associate uniquely a Lie algebra $(2m-1)$-cocycle. Conversely, we also have the following

Proposition 4.6
Let $\Omega^{(2m-1)}$ be a primitive cocycle. The l polynomials $t^{(m)}$ given by

$$t^{i_1\ldots i_m} = [\Omega^{(2m-1)}]^{j_1\ldots j_{2m-2} i_m} C^{i_1}_{j_1 j_2}\ldots C^{i_{m-1}}_{j_{2m-3}j_{2m-2}} \tag{4.19}$$

are invariant, symmetric and primitive (see [11, Lemma 3.2]).

This converse proposition relates the cocycles of the Lie algebra cohomology to Casimirs in the enveloping algebra $\mathcal{U}(\mathcal{G})$. The polynomials in (4.19) have certain advantages (for instance, they have all traces equal to zero) [11] over other more conventional ones such as *e.g.*, those in (4.5).

4.3 The case of simple compact groups

We have seen that the Lie algebra cocycles may be expressed in terms of LI forms on the group manifold G (Sec. 3.2). For compact groups, the CE cohomology can be identified (see, *e.g.* [1]) with the de Rham cohomology:

Proposition 4.7
Let G be a compact and connected Lie group. Every de Rham cohomology class on G contains one and only one bi-invariant form. The bi-invariant forms span a ring isomorphic to $H_{DR}(G)$.

The equivalence of the Lie algebra (CE) cohomology and the de Rham cohomology is specially interesting because, since all primitive cocycles are odd, compact groups behave as products of odd spheres from the point of view of real homology. This leads to a number of simple an elegant formulae concerning the Poincaré polynomials, Betti numbers, etc. We conclude by giving a table (table 4.1) which summarizes many of these results. Details on the topological properties of Lie groups may be found in [16, 17, 18, 19, 20, 21, 22]; for book references see [23, 24, 25, 2].

$\mathcal{G}$	$\dim \mathcal{G}$	order of invariants and Casimirs	order of $\mathcal{G}$-cocycles
A_l	$(l+1)^2-1$ $[l>1]$	$2,3,\ldots,l+1$	$3,5,\ldots,2l+1$
B_l	$l(2l+1)$ $[l>2]$	$2,4,\ldots,2l$	$3,7,\ldots,4l-1$
C_l	$l(2l+1)$ $[l>3]$	$2,4,\ldots,2l$	$3,7,\ldots,4l-1$
D_l	$l(2l-1)$ $[l>4]$	$2,4,\ldots,2l-2,l$	$3,7,\ldots,4l-5,2l-1$
G_2	14	$2,6$	$3,11$
F_4	52	$2,6,8,12$	$3,11,15,23$
E_6	78	$2,5,6,8,9,12$	$3,9,11,15,17,23$
E_7	133	$2,6,8,10,12,14,18$	$3,11,15,19,23,27,35$
E_8	248	$2,8,12,14,18,20,24,30$	$3,15,23,27,35,39,47,59$

Table 4.1: Order of the primitive invariant polynomials and associated cocycles for all the simple Lie algebras.

5 Higher order simple and SH Lie algebras

We present here a construction for which the previous cohomology notions play a crucial role, namely the construction of higher order Lie algebras. Recall that ordinary Lie algebras are defined as vector spaces endowed with the Lie bracket, which obeys the JI. If the Lie algebra is simple $\omega_{ij\rho} = k_{\rho\sigma}C^{\sigma}_{ij}$ is the non-trivial three-cocycle associated with the Cartan-Killing metric, given by the structure constant themselves (see (4.12)). The question arises as to whether higher order cocycles (and therefore Casimirs of order higher than two) can be used to define the structure constants of a higher order bracket. Given the odd-dimension of the cocycles, these multibrackets will involve an even number of Lie algebra elements. Since we already have matrix realizations of the simple Lie algebras, let us use them to construct the higher order brackets. Consider the case of $su(n)$, $n>2$ and a four-bracket. Let X_i be the matrices of the defining representation. Since the bracket has to be totally skew-symmetric, a sensible definition for it is

$$[X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}] := \varepsilon^{j_1 j_2 j_3 j_4}_{i_1 i_2 i_3 i_4} X_{j_1} X_{j_2} X_{j_3} X_{j_4} \quad . \tag{5.1}$$

This four-bracket generalizes the ordinary (two-) bracket $[X_{i_1}, X_{i_2}] = \varepsilon_{i_1 i_2}^{j_1 j_2} X_{j_1} X_{j_2}$. By using the skew-symmetry in $j_1 \dots j_4$, we may rewrite (5.1) in terms of commutators as

$$\begin{aligned}
[X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}] &= \frac{1}{2^2} \varepsilon_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} [X_{j_1}, X_{j_2}][X_{j_3}, X_{j_4}] = \frac{1}{2^2} \varepsilon_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} C_{j_1 j_2}^{l_1} C_{j_3 j_4}^{l_2} X_{l_1} X_{l_2} \\
&= \frac{1}{2^2} \varepsilon_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} C_{j_1 j_2}^{l_1} C_{j_3 j_4}^{l_2} \frac{1}{2} (d_{l_1 l_2 .}{}^{\sigma} X_\sigma + c \delta_{l_1 l_2}) \\
&= \frac{1}{2^3} \varepsilon_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} C_{j_1 j_2}^{l_1} C_{j_3 j_4}^{l_2} d_{l_1 l_2 .}{}^{\sigma} X_\sigma = \omega_{i_1 \dots i_4 .}{}^{\sigma} X_\sigma \quad ,
\end{aligned} \tag{5.2}$$

where in going from the first line to the second we have used that the factor multiplying $X_{l_1} X_{l_2}$ is symmetric in l_1, l_2, so that we can replace $X_{l_1} X_{l_2}$ by $\frac{1}{2}\{X_{l_1}, X_{l_2}\}$ and then write it in terms of the d's. The contribution of the term proportional to c vanishes due to the JI. Thus, the structure constants of the four-bracket are given by the 5-cocycle corresponding to the primitive polynomial d_{ijk}. These reasonings can be generalized to higher order brackets and to the other simple algebras. This motivates the following

Definition 5.1 (*Higher order bracket*)
Let X_i be arbitrary associative operators. The corresponding higher order bracket or multi-bracket of order n is defined by [26]

$$[X_1, \dots, X_n] := \sum_{\sigma \in S_n} (-1)^{\pi(\sigma)} X_{i_{\sigma(1)}} \dots X_{i_{\sigma(n)}} \quad . \tag{5.3}$$

The bracket (5.3) obviously satisfies the JI when $n = 2$. In the general case, the situation depends on whether n is even or odd, as stated by

Proposition 5.1
For n even, the n-bracket (5.3) satisfies the generalized Jacobi identity (GJI) [26]

$$\sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} \left[[X_{\sigma(1)}, \dots, X_{\sigma(n)}], X_{\sigma(n+1)}, \dots, X_{\sigma(2n-1)} \right] = 0 \quad ; \tag{5.4}$$

for n odd, the l.h.s. of (5.4) is proportional to $[X_1, \dots, X_{2n-1}]$.

Proof. In terms of the Levi-Civita symbol, the l.h.s. of (5.4) reads

$$\varepsilon_{i_1 \dots i_{2n-1}}^{j_1 \dots j_{2n-1}} \varepsilon_{j_1 \dots j_n}^{l_1 \dots l_n} [X_{l_1} \cdots X_{l_n}, X_{j_{n+1}}, \dots, X_{j_{2n-1}}] \quad . \tag{5.5}$$

Notice that the product $X_{l_1} \cdots X_{l_n}$ is a single entry in the n-bracket $[X_{l_1} \cdots X_{l_n}, X_{j_{n+1}}, \dots, X_{j_{2n-1}}]$. Since the n entries in this bracket are also antisymmetrized, eq. (5.5) is equal to

$$\begin{aligned}
& n! \varepsilon_{i_1 \dots \dots \dots i_{2n-1}}^{l_1 \dots l_n j_{n+1} \dots j_{2n-1}} \varepsilon_{j_{n+1} \dots j_{2n-1}}^{l_{n+1} \dots l_{2n-1}} \sum_{s=0}^{n-1} (-1)^s X_{l_{n+1}} \cdots X_{l_{n+s}} X_{l_1} \cdots X_{l_n} X_{l_{n+1+s}} \cdots X_{l_{2n-1}} \\
=& n!(n-1)! \varepsilon_{i_1 \dots i_{2n-1}}^{l_1 \dots l_{2n-1}} X_{l_1} \cdots X_{l_{2n-1}} \sum_{s=0}^{n-1} (-1)^s (-1)^{ns} \\
=& n!(n-1)! [X_{i_1}, \dots, X_{i_{2n-1}}] \sum_{s=0}^{n-1} (-1)^{s(n+1)} \quad ,
\end{aligned} \tag{5.6}$$

where we have used the skew-symmetry of ε to relocate the block $X_{l_1} \cdots X_{l_n}$ in the second equality. Thus, the l.h.s. of (5.4) is proportional to a multibracket of order $(2n-1)$ times a sum, which for even n vanishes and for odd n is equal to n, *q.e.d.*

In view of the above result, we introduce the following definition [26]

Definition 5.2 (*Higher order Lie algebra*)
An order n (n even) generalized Lie algebra is a vector space V of elements $X \in V$ endowed with a fully skew-symmetric bracket $V \times \overset{n}{\cdots} \times V \to V$, $(X_1, \ldots, X_n) \mapsto [X_1, \ldots, X_n] \in V$ such that the GJI (5.4) is fulfilled.

Consequently, a finite-dimensional Lie algebra of order $n = 2p$, generated by the elements $\{X_i\}_{i=1,\ldots,r}$ will be defined by an equation of the form

$$[X_{i_1}, \ldots, X_{i_{2p}}] = C_{i_1 \ldots i_{2p}}{}^{j} X_j \quad , \tag{5.7}$$

where $C_{i_1 \ldots i_{2p}}{}^{j}$ are the generalized structure constants. An example of this is provided by the construction given in (5.2), where the bracket is defined as in (5.3) and the structure constants are $(2p+1)$-cocycles of the simple Lie algebra used, $\Omega_{i_1 \ldots i_{2p}\sigma}$. Writing now the GJI (5.4) in terms of the Ω's, the following equation is obtained

$$\varepsilon^{j_1 \ldots j_{4p-1}}_{i_1 \ldots i_{4p-1}} \Omega_{j_1 \ldots j_{2p}}{}^{\sigma} \Omega_{\sigma j_{2p+1} \ldots j_{4p-1} \rho} = 0 \quad . \tag{5.8}$$

This equation is known to hold due to Proposition 5.1 and a generalization of the argument given in (5.2), which in fact provides the proof of

Theorem 5.1 (*Classification theorem for higher-order simple Lie algebras*)
Given a simple algebra $\mathcal{G}$ of rank l, there are $l-1$ $(2m_i - 2)$-higher-order simple Lie algebras associated with $\mathcal{G}$. They are given by the $l-1$ Lie algebra cocycles of order $2m_i - 1 > 3$ which may be obtained from the $l-1$ symmetric invariant polynomials on $\mathcal{G}$ of order $m_i > m_1 = 2$. The $m_1 = 2$ case (Killing metric) reproduces the original simple Lie algebra $\mathcal{G}$; for the other $l-1$ cases, the skew-symmetric $(2m_i - 2)$-commutators define an element of $\mathcal{G}$ by means of the $(2m_i - 1)$-cocycles. These higher-order structure constants (as the ordinary structure constants with all the indices written down) are fully antisymmetric cocycles and satisfy the GJI.

Proposition 5.2 (*Mixed order generalized Jacobi identity*)
Let m, n be even. We introduce the mixed order generalized Jacobi identity for even order multibrackets by

$$\varepsilon^{j_1 \ldots j_{n+m-1}} \left[[X_{j_1}, \ldots, X_{j_n}], \ldots, X_{j_{n+m-1}} \right] = 0 \quad . \tag{5.9}$$

Proof. Following the same reasonings of Proposition 5.1,

$$\begin{aligned}
&\varepsilon^{j_1 \ldots j_{n+m-1}}_{i_1 \ldots i_{n+m-1}} \varepsilon^{l_1 \ldots l_n}_{j_1 \ldots j_n} [X_{l_1} \cdots X_{l_n}, X_{j_{n+1}}, \ldots, X_{j_{n+m-1}}] \\
&\quad = n! \varepsilon^{l_1 \ldots l_n j_{n+1} \ldots j_{n+m-1}}_{i_1 \ldots \ldots \ldots i_{n+m-1}} \varepsilon^{l_{n+1} \ldots l_{n+m-1}}_{j_{n+1} \ldots j_{n+m-1}} \sum_{s=0}^{m-1} (-1)^s X_{l_{n+1}} \cdots X_{l_{n+s}} X_{l_1} \cdots X_{l_n} X_{l_{n+1+s}} \cdots X_{l_{n+m-1}} \\
&\quad = n!(m-1)! \varepsilon^{l_1 \ldots l_{n+m-1}}_{i_1 \ldots i_{n+m-1}} X_{l_1} \cdots X_{l_{n+m-1}} \sum_{s=0}^{m-1} (-1)^s (-1)^{ns} \\
&\quad = n!(m-1)! [X_{i_1}, \ldots, X_{i_{n+m-1}}] \sum_{s=0}^{m-1} (-1)^{(n+1)s} \quad ,
\end{aligned} \tag{5.10}$$

which is zero for n and m even. In contrast, if n and/or m are odd the sum $\sum_{s=0}^{m-1}(-1)^{(n+1)s}$ is different from zero (m if n is odd and 1 if n is even). In this case, the l.h.s. of (5.9) is proportional to the $(n+m-1)$-commutator $[X_{i_1},\ldots,X_{i_{n+m-1}}]$, *q.e.d.*

In particular, if n and m are the orders of higher order algebras, the identity (5.9) leads to (cf. (5.8))

$$\varepsilon^{i_1\ldots i_{n+m-1}}\Omega_{i_1\ldots i_n}{}^{\sigma}\Omega_{\sigma i_{n+1}\ldots i_{n+m-1}\rho}=0 \quad . \tag{5.11}$$

For $n=2$ and $[X_i,X_j]=C_{ij}^k X_k$, $[X_{i_1},\ldots,X_{i_m}]=\Omega_{i_1\ldots i_m}{}^k X_k$ eq. (5.11) gives

$$\varepsilon^{i_1\ldots i_{m+1}}C^{\sigma}_{i_1 i_2}\Omega_{\sigma i_3\ldots i_{m+1}\rho}=0 \quad , \tag{5.12}$$

which implies that $\Omega_{i_1\ldots i_{m+1}}$ is a cocycle, *i.e.*,

$$\varepsilon^{i_1\ldots i_{m+2}}C^{\sigma}_{i_1 i_2}\Omega_{\sigma i_3\ldots i_{m+1} i_{m+2}}=0 \quad . \tag{5.13}$$

Expression (5.13) follows from (5.12), simply antisymmetrizing the index ρ.

5.1 Multibrackets and coderivations

Higher-order brackets can be used to generalize the ordinary coderivation of multivectors.

Definition 5.3
Let $\{X_i\}$ be a basis of $\mathcal{G}$ given in terms of LIVF on G, and $\wedge^*(\mathcal{G})$ the exterior algebra of multivectors generated by them ($X_1\wedge\ldots\wedge X_q\equiv\varepsilon^{i_1\ldots i_q}_{1\ldots q}X_{i_1}\otimes\ldots\otimes X_{i_q}$). The exterior coderivation $\partial:\wedge^q\to\wedge^{q-1}$ is given by

$$\partial(X_1\wedge\ldots\wedge X_q)=\sum_{\substack{l=1\\ l<k}}^{q}(-1)^{l+k+1}[X_l,X_k]\wedge X_1\wedge\ldots\wedge\hat{X}_l\wedge\ldots\wedge\hat{X}_k\wedge\ldots\wedge X_q \quad . \tag{5.14}$$

This definition is analogous to that of the exterior derivative d, as given by (1.9) with its first term missing when one considers left-invariant forms (eq. (2.4)). As d, ∂ is nilpotent, $\partial^2=0$, due to the JI for the commutator.

In order to generalize (5.14), let us note that $\partial(X_1\wedge X_2)=[X_1,X_2]$, so that (5.14) can be interpreted as a formula that gives the action of ∂ on a q-vector in terms of that on a bivector. For this reason we may write ∂_2 for ∂ above. It is then natural to introduce an operator ∂_s that on a s-vector gives the multicommutator of order s. On an n-multivector its action is given by

Definition 5.4 (*Coderivation* ∂_s)
The general coderivation ∂_s of degree $-(s-1)$ (s even) $\partial_s:\wedge^n(G)\to\wedge^{n-(s-1)}(G)$ is defined by

$$\begin{aligned}
&\partial_s(X_1\wedge\ldots\wedge X_n):=\frac{1}{s!}\frac{1}{(n-s)!}\varepsilon^{i_1\ldots i_n}_{1\ldots n}\partial_s(X_{i_1}\wedge\ldots\wedge X_{i_s})\wedge X_{i_{s+1}}\wedge\ldots\wedge X_{i_n} \quad ,\\
&\partial_s\wedge^n(G)=0 \quad \text{for } s>n \quad ,\\
&\partial_s(X_1\wedge\ldots\wedge X_s)=[X_1,\ldots,X_s] \quad .
\end{aligned} \tag{5.15}$$

Proposition 5.3
The coderivation (5.15) is nilpotent, *i.e.*, $\partial_s^2 \equiv 0$.

Proof. Let n and s be such that $n-(s-1) \geq s$ (otherwise the statement is trivial). Then,

$$\begin{aligned}
&\partial_s \partial_s (X_1 \wedge \ldots \wedge X_n) \\
&= \frac{1}{s!}\frac{1}{(n-s)!}\varepsilon^{i_1 \ldots i_n}_{1 \ldots n}\varepsilon^{j_{s+1} \ldots j_n}_{i_{s+1} \ldots i_n}\{s\,[X_{j_{s+1}}, \ldots, X_{j_{2s-1}}, [X_{i_1}, \ldots, X_{i_s}]] \wedge X_{j_{2s}} \wedge \ldots X_{j_n} \\
&- (n-s)[X_{j_{s+1}}, \ldots, X_{j_{2s}}] \wedge [X_{i_1}, \ldots, X_{i_s}] \wedge X_{j_{2s+1}} \wedge \ldots \wedge X_{j_n}\} = 0 \quad .
\end{aligned} \tag{5.16}$$

The first term vanishes because s is even and is proportional to the GJI. The second one is also zero because the wedge product of the two s-brackets is antisymmetric while the resulting ε symbol is symmetric under the interchange $(i_1, \ldots i_s) \leftrightarrow (j_{s+1}, \ldots, j_{2s})$, *q.e.d.*

Remark. A derivation satisfies Leibniz's rule (see Proposition 5.5 below), which we may express as $d \circ m = m \circ (d \otimes 1 + 1 \otimes d)$ acting on the product m of two copies of the algebra. The coderivation satisfies the dual property $\Delta \circ \partial = (\partial \otimes 1 + 1 \otimes \partial) \circ \Delta$, where Δ is the 'coproduct'. The simplest example corresponds to

$$\begin{aligned}
(\Delta \circ \partial)(X_1 \wedge X_2) &= \Delta(\partial(X_1 \wedge X_2)) = \Delta[X_1, X_2] = [X_1, X_2] \wedge 1 + 1 \wedge [X_1, X_2] = \\
&= (\partial \otimes 1 + 1 \otimes \partial)(2X_1 \wedge 1 \wedge X_2 + X_1 \wedge X_2 \wedge 1 + 1 \wedge X_1 \wedge X_2)
\end{aligned} \tag{5.17}$$

since $\Delta(X_1 \wedge X_2) = \Delta X_1 \wedge X_2 + X_1 \wedge \Delta X_2$.

Let us now see how the nilpotency condition (or equivalently the GJI) looks like in the simplest cases.

Example 5.1
Consider $\partial \equiv \partial_2$. Then we have

$$\partial(X_1 \wedge X_2 \wedge X_3) = [X_1, X_2] \wedge X_3 - [X_1, X_3] \wedge X_2 + [X_2, X_3] \wedge X_1 \tag{5.18}$$

and

$$\partial^2(X_1 \wedge X_2 \wedge X_3) = [[X_1, X_2], X_3] - [[X_1, X_3], X_2] + [[X_2, X_3], X_1] = 0 \quad . \tag{5.19}$$

Example 5.2
When we move to $\partial \equiv \partial_4$, the number of terms grows very rapidly. The explicit expression for $\partial^2(X_{i_1} \wedge \ldots \wedge X_{i_7}) = 0$ (which, as we know, is equivalent to the GJI) is given in [27, eq. (32)] (note that the tenth term there should read $[[X_{i_1}, X_{i_2}, X_{i_6}, X_{i_7}], X_{i_3}, X_{i_4}, X_{i_5}]$). It contains $\binom{7}{3} = 35$ terms. In general, the GJI which follows from $\partial^2_{2m-2}(X_1 \wedge \ldots \wedge X_{4m-5}) = 0$ ($s = 2m-2$) contains $\binom{4m-5}{2m-1}$ different terms.

These higher order Lie algebras turn out to be a special example of the strongly homotopy (SH) Lie algebras [28, 29, 30]. These allow for violations of the generalized Jacobi identity, which are absent in our case (for the physical relevance of multialgebras, see the references in [28, 26]).

Definition 5.5 (*Strongly homotopy Lie algebras* [28])
A *SH Lie structure* on a vector space V is a collection of skew-symmetric linear maps l_n : $V \otimes \overset{n}{\cdots} \otimes V \to V$ such that

$$\sum_{i+j=n+1} \sum_{\sigma \in S_n} \frac{1}{(i-1)!} \frac{1}{j!} (-1)^{\pi(\sigma)} (-1)^{i(j-1)} \, l_i(l_j(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \ldots \otimes v_{\sigma(n)}) = 0 \quad . \tag{5.20}$$

For a general treatment of SH Lie algebras including v gradings see [28, 29, 30] and references therein. Note that $\frac{1}{(i-1)!}\frac{1}{j!}\sum_{\sigma \in S_n}$ is equivalent to the sum over the 'unshuffles', *i.e.*, over the permutations $\sigma \in S_n$ such that $\sigma(1) < \ldots < \sigma(j)$ and $\sigma(j+1) < \ldots < \sigma(n)$.

Example 5.3
For $n = 1$, eq. (5.20) just says that $l_1^2 = 0$ (l_1 is a differential). For $n = 2$, eq. (5.20) gives

$$-\frac{1}{2} l_1(l_2(v_1 \otimes v_2) - l_2(v_2 \otimes v_1)) + l_2(l_1(v_1) \otimes v_2 - l_1(v_2) \otimes v_1) = 0 \tag{5.21}$$

i.e., $l_1[v_1, v_2] = [l_1 v_1, v_2] + [v_1, l_1 v_2]$ with $l_2(v_1 \otimes v_2) = [v_1, v_2]$.

For $n = 3$, we have three maps l_1 , l_2 , l_3, and eq. (5.20) reduces to

$$\begin{aligned} &[l_2(l_2(v_1 \otimes v_2) \otimes v_3) + l_2(l_2(v_2 \otimes v_3) \otimes v_1) + l_2(l_2(v_3 \otimes v_1) \otimes v_2)] + [l_1(l_3(v_1 \otimes v_2 \otimes v_3))] \\ &\quad + [l_3(l_1(v_1) \otimes v_2 \otimes v_3) + l_3(l_1(v_2) \otimes v_3 \otimes v_1) + l_3(l_1(v_3) \otimes v_1 \otimes v_2)] = 0 \quad , \end{aligned} \tag{5.22}$$

i.e., adopting the convention that $l_n(v_1 \otimes \ldots \otimes v_n) = [v_1, \ldots, v_n]$,

$$\begin{aligned} &[[v_1, v_2], v_3] + [[v_2, v_3], v_1] + [[v_3, v_1], v_2] \\ &\quad = -l_1[v_1, v_2, v_3] - [l_1(v_1), v_2, v_3] - [v_1, l_1(v_2), v_3] - [v_1, v_2, l_1(v_3)] \quad . \end{aligned} \tag{5.23}$$

The second line in (5.23) shows the violation of the (standard) Jacobi identity given in the first line.

In the particular case in which a unique l_n (n even) is defined, we recover Def. 5.2 of a higher order Lie algebra since, for $i = j = n$ eq. (5.20) reproduces the GJI (5.4) in the form

$$\sum_{\sigma \in S_{2n-1}} \frac{1}{n!} \frac{1}{(n-1)!} (-1)^{\pi(\sigma)} l_n(l_n(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}) \otimes v_{\sigma(n+1)} \otimes \ldots \otimes v_{\sigma(2n-1)}) = 0 \quad . \tag{5.24}$$

We wish to conclude this subsection by pointing out that n-algebras have also been considered in [31, 32, 33].

5.2 The complete BRST operator for a simple Lie algebra

We now generalize the BRST operator and MC equations of Sec. 3.4 to the general case of higher-order simple Lie algebras. The result is a new BRST-type operator that contains the information of all the l possible algebras associated with a given simple Lie algebra $\mathcal{G}$ of rank l.

Let us first note that, in the notation of (2.6), the JI reads

$$d^2\theta = -d(\theta\wedge\theta) = \frac{1}{2}\,[[\theta,\theta],\theta] = 0 \quad , \tag{5.25}$$

and expresses the nilpotency of d. Now, in Sec. 5.1 we considered higher-order coderivations which also had the property $\partial_s^2 = 0$ as a result of the GJI. We may now introduce the corresponding dual higher-order derivations $\tilde{d}_s$ to provide a generalization of the Maurer-Cartan equations (2.3). Since ∂_s was defined on multivectors that are product of left-invariant vector fields, the dual $\tilde{d}_s$ will be given for left-invariant forms.

It is easy to introduce dual basis in $\wedge_n$ and $\wedge^n$. With $\omega^i(X_j) = \delta^i_j$, a pair of dual basis in $\wedge_n$, $\wedge^n$ are given by $\omega^{I_1}\wedge\ldots\wedge\omega^{I_n}$, $\frac{1}{n!}X_{I_1}\wedge\ldots\wedge X_{I_n}$ $(I_1<\ldots<I_n)$ since $(\varepsilon^{i_1\ldots i_n}_{j_1\ldots j_n}\omega^{j_1}\otimes\ldots\otimes\omega^{j_n})(\frac{1}{n!}\varepsilon^{k_1\ldots k_n}_{l_1\ldots l_n}X_{k_1}\otimes\ldots\otimes X_{k_n}) = \varepsilon^{i_1\ldots i_n}_{l_1\ldots l_n}$ and $\varepsilon^{I_1\ldots I_n}_{L_1\ldots L_n}$ is 1 if all indices coincide and 0 otherwise. Nevertheless it is customary to use the non-minimal set $\omega^{i_1}\wedge\ldots\wedge\omega^{i_n}$ to write $\alpha = \frac{1}{n!}\alpha_{i_1\ldots i_n}\omega^{i_1}\wedge\ldots\wedge\omega^{i_n}$. Since $(\omega^{i_1}\wedge\ldots\wedge\omega^{i_n})(X_{j_1},\ldots,X_{j_n}) = \varepsilon^{i_1\ldots i_n}_{j_1\ldots j_n}$ it is clear that $\alpha_{i_1\ldots i_n} = \alpha(X_{i_1},\ldots,X_{i_n}) = \frac{1}{n!}\alpha(X_{i_1}\wedge\ldots\wedge X_{i_n})$.

Definition 5.6
The action of $\tilde{d}_m : \wedge_n \to \wedge_{n+(2m-3)}$ (remember that $s=2m-2$) on $\alpha\in\wedge_n$ is given by (cf. (2.5))

$$\begin{aligned}
&(\tilde{d}_m\alpha)(X_{i_1},\ldots,X_{i_{n+2m-3}}) := \\
&\qquad \frac{1}{(2m-2)!}\frac{1}{(n-1)!}\varepsilon^{j_1\ldots j_{n+2m-3}}_{i_1\ldots i_{n+2m-3}}\alpha([X_{j_1},\ldots,X_{j_{2m-2}}],X_{j_{2m-1}},\ldots,X_{j_{n+2m-3}}) \quad , \\
&(\tilde{d}_m\alpha)_{i_1\ldots i_{n+2m-3}} = \frac{1}{(2m-2)!}\frac{1}{(n-1)!}\varepsilon^{j_1\ldots j_{n+2m-3}}_{i_1\ldots i_{n+2m-3}}\Omega_{j_1\ldots j_{2m-2}\cdot}{}^{\rho}\alpha_{\rho j_{2m-1}\ldots j_{n+2m-3}} \quad .
\end{aligned} \tag{5.26}$$

Proposition 5.4
$\tilde{d}_m$ is dual to the coderivation $\partial_{2m-2} : \wedge^n \to \wedge^{n-(2m-3)}$, $(\tilde{d}_2 = -d,\ \tilde{d}_2 : \wedge_n \to \wedge_{n+1})$.

Proof. We have to check the 'duality' relation $\tilde{d}_m\alpha \propto \alpha\partial_{2m-2}$ $(\partial_{2m-2} : \wedge_{n+(2m-3)} \to \wedge_n)$. Indeed, if α is an n-form, eq. (5.15) tells us that

$$\begin{aligned}
\alpha\left(\partial_{2m-2}(X_{i_1}\wedge\ldots\wedge X_{i_{n+2m-3}})\right) &= \frac{1}{(2m-2)!}\frac{1}{(n+2m-3-2m+2)!}\times \\
&\times\varepsilon^{j_1\ldots j_{n+2m-3}}_{i_1\ldots i_{n+2m-3}}\alpha([X_{j_1},\ldots,X_{j_{2m-2}}]\wedge X_{j_{2m-1}}\wedge\ldots\wedge X_{j_{n+2m-3}}) \quad ,
\end{aligned} \tag{5.27}$$

which is proportional[5] to $(\tilde{d}_m\alpha)(X_{i_1}\wedge\ldots\wedge X_{i_{n+2m-3}})$, *q.e.d.*

Proposition 5.5
The operator $\tilde{d}_m$ satisfies Leibniz's rule.

[5] One finds $\tilde{d}_m\alpha = \frac{(n+2m-3)!}{n!}\alpha\partial_{2m-2}$, where n is the order of the form α. The factor appears as a consequence of using the same definition (antisymmetrization with no weight factor) for the $\wedge$ product of forms and vectors.

Proof. For $\alpha \in \wedge_n$, $\beta \in \wedge_p$ we get, using (5.26)

$$
\begin{aligned}
\tilde{d}_m(\alpha\wedge\beta)_{i_1\ldots i_{n+p+2m-3}} &= \frac{1}{(2m-2)!}\frac{1}{(n+p-1)}\varepsilon^{j_1\ldots j_{n+p+2m-3}}_{i_1\ldots i_{n+p+2m-3}}\Omega_{j_1\ldots j_{2m-2}\cdot}{}^{\rho} \\
&\quad\cdot\Big(\frac{1}{n!p!}\varepsilon^{k_1\ldots\ldots\ldots k_{n+p}}_{\rho j_{2m-1}\ldots j_{n+p+2m-3}}\alpha_{k_1\ldots k_n}\beta_{k_{n+1}\ldots k_{n+p}}\Big) \\
&= \frac{1}{(2m-2)!}\frac{1}{n!p!}\varepsilon^{j_1\ldots j_{n+p+2m-3}}_{i_1\ldots i_{n+p+2m-3}}\Omega_{j_1\ldots j_{2m-2}\cdot}{}^{\rho}\Big(n\alpha_{\rho j_{2m-1}\ldots j_{n+2m-3}}\beta_{j_{n+2m-2}\ldots j_{n+p+2m-3}} \\
&\quad +(-1)^n p\alpha_{j_{2m-1}\ldots j_{n+2m-2}}\beta_{\rho j_{n+2m-1}\ldots j_{n+p+2m-3}}\Big) \\
&= \varepsilon^{j_1\ldots j_{n+p+2m-3}}_{i_1\ldots i_{n+p+2m-3}}\Big(\frac{1}{p!(n+2m-3)!}(\tilde{d}_m\alpha)_{j_1\ldots j_{n+2m-3}}\beta_{j_{n+2m-2}\ldots j_{n+p+2m-3}} \\
&\quad +(-1)^n\frac{1}{n!(p+2m-3)!}\alpha_{j_{2m-1}\ldots j_{n+2m-2}}(\tilde{d}_m\beta)_{j_1\ldots j_{2m-2}j_{n+2m-1}\ldots j_{n+p+2m-3}}\Big) \\
&= \Big((\tilde{d}_m\alpha)\wedge\beta+(-1)^n\alpha\wedge(\tilde{d}_m\beta)\Big)_{i_1\ldots i_{n+p+2m-3}} \quad .
\end{aligned}
\tag{5.28}
$$

Thus, $\tilde{d}_m$ is odd and $\tilde{d}_m(\alpha\wedge\beta)=\tilde{d}_m\alpha\wedge\beta+(-1)^n\alpha\wedge\tilde{d}_m\beta$, *q.e.d.*

The coordinates of $\tilde{d}_m\omega^\sigma$ are given by

$$
\begin{aligned}
(\tilde{d}_m\omega^\sigma)(X_{i_1},\ldots,X_{i_{2m-2}}) &= \frac{1}{(2m-2)!}\varepsilon^{j_1\ldots j_{2m-2}}_{i_1\ldots i_{2m-2}}\omega^\sigma([X_{j_1},\ldots,X_{j_{2m-2}}]) \\
&= \omega^\sigma([X_{i_1},\ldots,X_{i_{2m-2}}]) = \omega^\sigma(\Omega_{i_1\ldots i_{2m-2}\cdot}{}^{\rho}X_\rho) = \Omega_{i_1\ldots i_{2m-2}\cdot}{}^{\sigma}
\end{aligned}
\tag{5.29}
$$

from which we conclude that

$$
\tilde{d}_m\omega^\sigma = \frac{1}{(2m-2)!}\Omega_{i_1\ldots i_{2m-2}\cdot}{}^{\sigma}\omega^{i_1}\wedge\ldots\wedge\omega^{i_{2m-2}} \quad . \tag{5.30}
$$

For $m=2$, $\tilde{d}_2=-d$, equations (5.30) reproduce the MC eqs. (2.6). In the compact notation that uses the canonical one-form θ, we may now introduce the following

Proposition 5.6 (*Generalized Maurer-Cartan equations***)**
The action of $\tilde{d}_m$ on the canonical form θ is given by

$$
\tilde{d}_m\theta = \frac{1}{(2m-2)!}\left[\theta,\overset{2m-2}{\cdots},\theta\right] \quad , \tag{5.31}
$$

where the multibracket of forms is defined by $\left[\theta,\overset{2m-2}{\cdots},\theta\right]=\omega^{i_1}\wedge\ldots\wedge\omega^{i_{2m-2}}[X_{i_1},\ldots,X_{i_{2m-2}}]$.
Using Leibniz's rule for the operator $\tilde{d}_m$ we arrive at

$$
\tilde{d}_m^2\theta = -\frac{1}{(2m-2)!}\frac{1}{(2m-3)!}\left[\theta,\overset{2m-3}{\cdots},\theta,\left[\theta,\overset{2m-2}{\cdots},\theta\right]\right]=0 \quad , \tag{5.32}
$$

which again expresses the GJI.

Each Maurer-Cartan-like equation (5.32) can be expressed in terms of the ghost variables introduced in Sec. 3.4 by means of a 'generalized BRST operator',

$$
s_{2m-2} = -\frac{1}{(2m-2)!}c^{i_1}\ldots c^{i_{2m-2}}\Omega_{i_1\ldots i_{2m-2}\cdot}{}^{\sigma}\frac{\partial}{\partial c^\sigma} \quad . \tag{5.33}
$$

By adding together all the l generalized BRST operators, the complete BRST operator is obtained. Then we have the following

Theorem 5.2 (*Complete BRST operator*)
Let $\mathcal{G}$ be a simple Lie algebra. Then, there exists a nilpotent associated operator, the complete BRST operator associated with $\mathcal{G}$, given by the odd vector field

$$\begin{aligned} s = & -\frac{1}{2} c^{j_1} c^{j_2} \Omega_{j_1 j_2 \cdot}{}^{\sigma} \frac{\partial}{\partial c^{\sigma}} - \ldots - \frac{1}{(2m_i - 2)!} c^{j_1} \ldots c^{j_{2m_i-2}} \Omega_{j_1 \ldots j_{2m_i-2} \cdot}{}^{\sigma} \frac{\partial}{\partial c^{\sigma}} - \ldots \\ & - \frac{1}{(2m_l - 2)!} c^{j_1} \ldots c^{j_{2m_l-2}} \Omega_{j_1 \ldots j_{2m_l-2} \cdot}{}^{\sigma} \frac{\partial}{\partial c^{\sigma}} \equiv s_2 + \ldots + s_{2m_i-2} + \ldots + s_{2m_l-2} \quad , \end{aligned} \tag{5.34}$$

where $i = 1, \ldots, l$, $\Omega_{j_1 j_2 \cdot}{}^{\sigma} \equiv C_{j_1 j_2 \cdot}{}^{\sigma}$ and $\Omega_{j_1 \ldots j_{2m_i-2} \cdot}{}^{\sigma}$ are the corresponding l higher-order cocycles.

Proof. We have to show that $\{s_{2m_i-2}, s_{2m_j-2}\} = 0 \ \forall\, i, j$. To prove it, let us write the anti-commutator explicitly:

$$\begin{aligned} \{s_{2m_i-2}, s_{2m_j-2}\} = & \frac{1}{(2m_i-2)!} \frac{1}{(2m_j-2)!} \times \\ & \times \{(2m_j - 2) c^{l_1} \ldots c^{l_{2m_i-2}} \Omega_{l_1 \ldots l_{2m_i-2} \cdot}{}^{\rho} c^{r_2} \ldots c^{r_{2m_j-2}} \Omega_{\rho r_2 \ldots r_{2m_j-2} \cdot}{}^{\sigma} \frac{\partial}{\partial c^{\sigma}} + i \leftrightarrow j \\ & + (c^{l_1} \ldots c^{l_{2m_i-2}} c^{r_1} \ldots c^{r_{2m_j-2}} \Omega_{l_1 \ldots l_{2m_i-2} \cdot}{}^{\rho} \Omega_{r_1 \ldots r_{2m_j-2} \cdot}{}^{\sigma} + i \leftrightarrow j) \frac{\partial}{\partial c^{\rho}} \frac{\partial}{\partial c^{\sigma}} \} \\ = & \frac{1}{(2m_i-2)!} \frac{1}{(2m_j-3)!} c^{l_1} \ldots c^{l_{2m_i-2}} c^{r_2} \ldots c^{r_{2m_j-2}} \Omega_{l_1 \ldots l_{2m_i-2} \cdot}{}^{\rho} \Omega_{\rho r_2 \ldots r_{2m_j-2} \cdot}{}^{\sigma} \frac{\partial}{\partial c^{\sigma}} + i \leftrightarrow j \ , \end{aligned} \tag{5.35}$$

where we have used the fact that $\frac{\partial}{\partial c^{\rho}} \frac{\partial}{\partial c^{\sigma}}$ is antisymmetric in ρ, σ while the parenthesis multiplying it is symmetric. The term proportional to a single $\frac{\partial}{\partial c^{\sigma}}$ also vanishes as a consequence of equation (5.11), *q.e.d.*

The coefficients of $\partial / \partial c^{\sigma}$ in s_{2m_i-2} can be viewed, in dual terms, as (even) multivectors of the type

$$\Lambda = \frac{1}{(2m-2)!} \Omega_{i_1 \ldots i_{2m-2} \cdot}{}^{\sigma} x_{\sigma} \partial^{i_1} \wedge \ldots \wedge \partial^{i_{2m-2}} \quad . \tag{5.36}$$

(see (6.23)). They have the property of having zero Schouten-Nijenhuis bracket among themselves by virtue of the GJI (5.8).

Definition 5.7
Let us consider the algebra $\wedge(M)$ of multivectors on M. The Schouten-Nijenhuis bracket (SNB) of $A \in \wedge^p(M)$ and $B \in \wedge^q(M)$ is the unique extension of the Lie bracket of two vector fields to a bilinear mapping $\wedge^p(M) \times \wedge^q(M) \to \wedge^{p+q-1}(M)$ in such a way that $\wedge(M)$ becomes a graded superalgebra.

For the expression of the SNB in coordinates we refer to [34, 35]. It turns out that the multivector algebra with the exterior product and the SNB is a Gerstenhaber algebra[6], in which $\deg(A) = p - 1$ if $A \in \wedge^p$. Thus, the multivectors of the form (5.36) form an abelian subalgebra of this Gerstenhaber algebra, the commutativity (in the sense of the SNB) being a consequence of (5.8).

[6] A Gerstenhaber algebra [36] is a $\mathbb{Z}$-graded vector space (with homogeneous subspaces $\wedge^a$, a being the grade) with two bilinear multiplication operators, $\cdot$ and $[\ ,\ ]$ with the following properties ($u \in \wedge^a$, $v \in \wedge^b$,

6 Higher order generalized Poisson structures

We shall consider in this section two possible generalizations of the ordinary Poisson structures (PS) by brackets of more than two functions. The first one is the Nambu-Poisson structure (N-P) [38, 39, 40, 41] (see also [42]). The second, named generalized PS (GPS) [43, 44], is based on the previous constructions (and has been extended to the supersymmetric case [45]). We shall present both generalizations as well as examples of the GPS, which are naturally obtained from the higher-order simple Lie algebras of Sec. 5. A comparison between both structures may be found in [27] and in table 6.1 (see also [46]). Let us first review briefly the standard PS.

6.1 Standard Poisson structures

Definition 6.1

Let M be a differentiable manifold. A Poisson bracket (PB) on $\mathcal{F}(M)$ is a bilinear mapping $\{\cdot,\cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$ that satisfies ($f, g, h \in \mathcal{F}(M)$)

a) Skew-symmetry

$$\{f, g\} = -\{g, f\} \quad , \tag{6.1}$$

b) Leibniz's rule,

$$\{f, gh\} = g\{f, h\} + \{f, g\}h \quad , \tag{6.2}$$

c) Jacobi identity

$$\text{Alt}\{f, \{g, h\}\} = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \quad . \tag{6.3}$$

A PB on M defines a PS.

In local coordinates $\{x^i\}$, conditions a), b) and c) mean that it is possible to write

$$\{f(x), g(x)\} = \omega^{ij}\partial_i f \partial_j g \quad , \quad \omega^{ij} = -\omega^{ji} \quad , \quad \omega^{jk}\partial_k\omega^{lm} + \omega^{lk}\partial_k\omega^{mj} + \omega^{mk}\partial_k\omega^{jl} = 0 \quad . \tag{6.4}$$

It is possible to rewrite a)-c) in a geometrical way by using the bivector

$$\Lambda = \frac{1}{2}\omega^{jk}\partial_j \wedge \partial_k \quad , \tag{6.5}$$

in terms of which

$$\{f, g\} = \Lambda(df, dg) \quad ; \tag{6.6}$$

the JI imposes a condition on Λ, which is equivalent to the vanishing of the SNB [35]

$$[\Lambda, \Lambda] = 0 \quad . \tag{6.7}$$

$w \in \wedge^c$):

a) $\deg(u \cdot v) = a + b$,
b) $\deg[u, v] = a + b - 1$,
c) $(u \cdot v) \cdot w = u \cdot (v \cdot w)$,
d) $[u, v] = -(-1)^{(a-1)(b-1)}[v, u]$,
e) $(-1)^{(a-1)(c-1)}[u, [v, w]] + (-1)^{(c-1)(b-1)}[w, [u, v]] + (-1)^{(b-1)(a-1)}[v, [w, u]] = 0$,
f) $[u, v \cdot w] = [u, v] \cdot w + (-1)^{(a-1)b} v \cdot [u, w]$.

For an analysis of various related algebras, including Poisson algebras, see [37] and references therein.

If the manifold M is the dual of a Lie algebra, there always exists a PS, the Lie-Poisson structure, which is obtained by defining the fundamental Poisson bracket $\{x_i, x_j\}$ (where $\{x_i\}$ are coordinates on $\mathcal{G}^*$). Since $\mathcal{G} \sim (\mathcal{G}^*)^*$, we may think of $\mathcal{G}$ as a subspace of the ring of smooth functions $\mathcal{F}(\mathcal{G}^*)$. Then, the Lie algebra commutation relations

$$\{x_i, x_j\} = C_{ij}^k x_k \tag{6.8}$$

define, by assuming b) above, a mapping $\mathcal{F}(\mathcal{G}^*) \times \mathcal{F}(\mathcal{G}^*) \to \mathcal{F}(\mathcal{G}^*)$ associated with the bivector $\Lambda = \frac{1}{2} C_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_i}$. This is a PB since condition (6.4) (or (6.7)) is equivalent to the JI for the structure constants of $\mathcal{G}$.

6.2 Nambu-Poisson structures

Already in 1973, Nambu [38] considered the possibility of extending Poisson brackets to brackets of three functions. His attempt has been generalized since then, and all generalizations considered share the following two properties

$$\begin{array}{l} \text{a) } \{f_1, \ldots, f_i, \ldots, f_j, \ldots, f_n\} = -\{f_1, \ldots, f_j, \ldots, f_i, \ldots, f_n\} \text{ (skew-symmetry) ,} \\ \text{b) } \{f_1, \ldots, f_{n-1}, gh\} = g\{f_1, \ldots, f_{n-1}, h\} + \{f_1, \ldots, f_{n-1}, g\}h \text{ (Leibniz's rule)} \end{array} \tag{6.9}$$

which will be guaranteed if the bracket is generated in local coordinates $\{x_i\}$ on M by

$$\Lambda = \frac{1}{n!} \eta_{i_1 \ldots i_n} \partial^{i_1} \wedge \ldots \wedge \partial^{i_n} \tag{6.10}$$

as in (6.6), *i.e.* by

$$\{f_1, \ldots, f_n\} = \Lambda(df_1, \ldots, df_n) \quad . \tag{6.11}$$

The key difference among the higher order PS is the identity that generalizes c) in Definition 6.1. That corresponding to Nambu's mechanics was given by Sahoo and Valsakumar [39] and in the general case by Takhtajan [40], who studied it in detail and named it the *fundamental identity* (FI)

$$\begin{aligned} \{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} &= \{\{f_1, \ldots, f_{n-1}, g_1\}, g_2, \ldots, g_n\} \\ &+ \{g_1, \{f_1, \ldots, f_{n-1}, g_2\}, g_3, \ldots, g_n\} + \ldots + \{g_1 \ldots, g_{n-1}, \{f_1, \ldots, f_{n-1}, g_n\}\} \end{aligned} \tag{6.12}$$

(see also [41, 42]). The FI (6.12), together with (6.9), define the Nambu-Poisson structures [40]. To see the signification of (6.12), let us consider $n-1$ 'Hamiltonians' $(H_1, \ldots, H_{n-1})$ and define the time evolution of an observable by

$$\dot{g} = \{H_1, \ldots, H_{n-1}, g\} \quad . \tag{6.13}$$

Then, the FI guarantees that

$$\frac{d}{dt}\{g_1, \ldots, g_n\} = \{\dot{g}_1, \ldots, g_n\} + \ldots + \{g_1, \ldots, \dot{g}_n\} \quad , \tag{6.14}$$

i.e., that the time derivative is a derivation of the N-P n-bracket. In this way, the bracket of any n constants of the motion is itself a constant of the motion.

Inserting (6.10) into (6.12), one gets two conditions [40] for the coordinates $\eta_{i_1 \ldots i_n}$ of Λ. The first is the *differential condition*, which in local coordinates may be written as

$$\eta_{i_1 \ldots i_{n-1} \rho} \partial^\rho \eta_{j_1 \ldots j_n} - \frac{1}{(n-1)!} \epsilon_{j_1 \ldots j_n}^{l_1 \ldots l_n} (\partial^\rho \eta_{i_1 \ldots i_{n-1} l_1}) \eta_{\rho l_2 \ldots l_n} = 0 \quad . \tag{6.15}$$

The second is the *algebraic condition*. It follows from requiring the vanishing of the second derivatives in (6.12). In local coordinates it reads

$$\Sigma + P(\Sigma) = 0 \quad , \tag{6.16}$$

where Σ is the $2n$-tensor

$$\begin{aligned} \Sigma_{i_1\ldots i_n j_1 \ldots j_n} = \; & \eta_{i_1\ldots i_n}\eta_{j_1\ldots j_n} - \eta_{i_1\ldots i_{n-1}j_1}\eta_{i_n j_2\ldots j_n} - \eta_{i_1\ldots i_{n-1}j_2}\eta_{j_1 i_n j_3\ldots j_n} \\ - \; & \eta_{i_1\ldots i_{n-1}j_3}\eta_{j_1 j_2 i_n j_4\ldots j_n} - \ldots - \eta_{i_1\ldots i_{n-1}j_n}\eta_{j_1 j_2\ldots j_{n-1} i_n} \; . \end{aligned} \tag{6.17}$$

It turns out [47, 48, 49] (see also [50]) that this last condition implies that Λ in (6.10) is decomposable, *i.e.*, that Λ can be written as the exterior product of vector fields.

6.3 Generalized Poisson structures

Instead of generalizing Jacobi's identity through the FI (6.12), one may take a different path by following a geometrical rather than a dynamical approach. Since for the ordinary PS the JI is given by (6.7), it is natural [43, 44] to introduce in the even case new GPS by means of

Definition 6.2
A $2p$-multivector $\Lambda^{(2p)}$ defines a GPS if it satisfies

$$[\Lambda^{(2p)}, \Lambda^{(2p)}] = 0 \quad , \tag{6.18}$$

where $[\ ,\ ]$ denotes again the SNB. Notice that for n odd, $[\Lambda^{(n)}, \Lambda^{(n)}]$ vanishes identically and hence the condition is empty. Written in terms of the coordinates of $\Lambda^{(2p)}$ (now denoted $\omega_{i_1\ldots i_{2p}}$), the GJI condition (6.18) reads

$$\varepsilon^{j_1\ldots j_{4p-1}}_{i_1\ldots i_{4p-1}} \omega_{j_1\ldots j_{2p-1}\sigma} \partial^\sigma \omega_{j_{2p}\ldots j_{4p-1}} = 0 \tag{6.19}$$

[cf. (5.8)]. Thus a GPS is defined by (6.9) and eq. (6.18) (or (6.19)), which in terms of the GPB is expressed by

Proposition 6.1
The GJI for the GPB

$$\begin{aligned} & \mathrm{Alt}\{f_1, \ldots, f_{2p-1}, \{f_{2p}, \ldots, f_{4p-1}\}\} \\ & \quad := \sum_{\sigma \in S_{4p-1}} (-1)^{\pi(\sigma)} \{f_{\sigma(1)}, \ldots, f_{\sigma(2p-1)}, \{f_{\sigma(2p)}, \ldots, f_{\sigma(4p-1)}\}\} = 0 \quad , \end{aligned} \tag{6.20}$$

is equivalent [43, 44] to condition (6.19).

Proof. Let us write (6.20) as

$$\begin{aligned} & \varepsilon^{j_1\ldots j_{4p-1}}_{i_1\ldots i_{4p-1}} \{f_{j_1}, \ldots, f_{j_{2p-1}}, \omega_{l_{2p}\ldots l_{4p-1}} \partial^{l_{2p}} f_{j_{2p}} \ldots \partial^{l_{4p-1}} f_{j_{4p-1}}\} \\ & \quad = \varepsilon^{j_1\ldots j_{4p-1}}_{i_1\ldots i_{4p-1}} \omega_{l_1\ldots l_{2p-1}\sigma} \partial^{l_1} f_{j_1} \ldots \partial^{l_{2p-1}} f_{j_{2p-1}} (\partial^\sigma \omega_{l_{2p}\ldots l_{4p-1}} \partial^{l_{2p}} f_{j_{2p}} \ldots \partial^{l_{4p-1}} f_{j_{4p-1}} \\ & \quad + 2p\,\omega_{l_{2p}\ldots l_{4p-1}} \partial^\sigma \partial^{2p} f_{j_{2p}} \partial^{l_{2p+1}} f_{j_{2p+1}} \ldots \partial^{l_{4p-1}} f_{j_{4p-1}}) = 0 \quad . \end{aligned} \tag{6.21}$$

The second term vanishes because the factor multiplying $\partial^\sigma \partial^{2p} f_{j_{2p}}$ is antisymmetric with respect to the interchange $\sigma \leftrightarrow l_{2p}$. Hence, we are left with (6.19) because $f_{j_1}, \ldots, f_{j_{4p-1}}$ are arbitrary, *q.e.d.*

Remarks. 1. It is also possible to define the GPS in the *odd* case [27]. For GPB with an odd number of arguments, the second term in (6.21) does not vanish, giving now rise (as for the N-P structures) to an 'algebraic condition' which is absent in the even case [27].
2. These constructions may also be extended to the $\mathbb{Z}_2$-graded ('supersymmetric') case [45].
3. The GJI does not imply the FI. Thus, the GPB of constants of the motion is not a constant of the motion in general (see [44], however, for a weaker result). On the other hand, the FI does imply the GJI when n is even (and also when n is odd). So the GPS's may be viewed as a generalization of the Nambu-Takhtajan one. As a result, a Λ defining a GPS is not decomposable in general.

6.4 Higher order linear Poisson structures

It is now easy to construct examples of GPS (infinitely many, in fact) in the linear case. They are obtained by extending the argument at the end of Sec. 6.1 to the GPS. Let $\mathcal{G}$ be a simple Lie algebra of rank l. We know from Sec. 5 that corresponding to it there are $(l-1)$ higher order Lie algebras. Their structure constants define a GPB $\{\cdot,\overset{2m_l-2}{\cdots},\cdot\}:\mathcal{G}^*\times\overset{2m_l-2}{\cdots}\times\mathcal{G}^*\to\mathcal{G}^*$ by

$$\{x_{i_1},\ldots,x_{i_{2m_l-2}}\}=\Omega_{i_1\ldots i_{2m_l-2}.}{}^{\sigma}x_\sigma \quad , \tag{6.22}$$

where Ω is the $(2m_l-1)$-cocycle. If one now computes the GJI (6.19) for $\omega_{i_1\ldots i_{2m_l-2}}=\Omega_{i_1\ldots i_{2m_l-2}.}{}^{\sigma}x_\sigma$, or, alternatively, $[\Lambda,\Lambda]$ for

$$\Lambda=\frac{1}{(2m-2)!}\Omega_{i_1\ldots i_{2m-2}.}{}^{\sigma}x_\sigma\partial^{i_1}\wedge\ldots\wedge\partial^{i_{2m-2}} \quad , \tag{6.23}$$

one sees that $[\Lambda,\Lambda]=0$ is satisfied since it expresses the GJI for the higher order structure constants Ω given in (5.8). This means that *all* higher-order simple Lie algebras define linear GPS. These structures are not of the Nambu-Poisson type.

Conversely, given a linear GPS with fundamental GPB (6.22), the associated higher-order Lie algebra provides a realization of it. This is what one might expect to achieve when quantizing the classical theory if, that is, quantization implies the replacement of observables by *associative* operators and the GPB by multicommutators (the standard quantization *à la* Dirac implies the well known substitution $\{\ ,\ \}\mapsto\frac{1}{i\hbar}[\ ,\ ]$). The physical difficulty for the GPS is the fact that time derivative is not a derivation of the bracket (Sec. 6.3). The N-P structures are free from this problem, but the FI is not an identity for the algebra of associative operators. Thus, one is led to the conclusion that a standard quantization of higher order mechanics is not possible (see, however, [50]) and that ordinary Hamiltonian mechanics is, in this sense, rather unique.

7 Relative cohomology, coset spaces and effective WZW actions

This is a topic of recent physical interest [51, 52, 53] since, for an action invariant under the compact symmetry group G which has a vacuum that is symmetric under the subgroup H, the Goldstone fields parametrize the coset space. Thus, the possible invariant effective actions of WZW type [13, 14] are related with the cohomology in G/H. In particular, for the cohomology of degree 4 and 5 we may construct WZW actions on 3- and 4- dimensional space-times respectively.

	PS	N-P	GPS (even order)
Characteristic identity (CI):	Eq. (6.3) (JI)	Eq. (6.12) (FI)	Eq. (6.20) (GJI)
Defining conditions:	Eq. (6.4)	Eqs. (6.15),(6.16)	Eq. (6.19)
Liouville theorem:	Yes	Yes	Yes
Poisson theorem:	Yes	Yes	No (in general)
CI realization in terms of associative operators:	Yes	No (in general)	Yes

Table 6.1: Some properties of Nambu-Poisson (N-P) and generalized Poisson (GP) structures.

Let G be a compact Lie group and H a subgroup. The 'left coset' $K = G/H$ is defined through the projection map $\pi : G \to K$ by

$$\pi : gh \to \{gH\} \equiv g \quad , \quad \forall\, h \in H \quad . \tag{7.1}$$

$G(H,K)$ is a principal bundle where the structure group H acts on the right $R_h : g \mapsto gh$ and the base space is the coset G/H.

Theorem 7.1 (*Projectable forms*)
Let $G(H,K)$ be a principal bundle. A q-form Ω on G is projectable to a form $\bar{\Omega}$ on K, *i.e.*, there exists a unique $\bar{\Omega}$ such that $\Omega = \pi^*(\bar{\Omega})$ iff

$\Omega(g)(X_1(g), \ldots, X_q(g)) = 0$ if one $X \in \mathfrak{X}(H)$ (Ω is horizontal)

$R_h^*\Omega = \Omega$ (Ω is invariant under the right action of H).

Proof. See [54].

Definition 7.1 (*Relative Lie algebra cohomology*)
Let $\mathcal{G}$ be a Lie algebra and $\mathcal{H}$ a subalgebra of $\mathcal{G}$. The space of relative (to the subalgebra $\mathcal{H}$) q-cochains $C^q(\mathcal{G},\mathcal{H})$ is that of the q-skew-symmetric maps $\Omega : \mathcal{G} \wedge \overset{q}{\cdots} \wedge \mathcal{G} \to \mathbb{R}$ such that (cf. Theorem 7.1)

$$\begin{aligned} &\Omega(X, X_2, \ldots, X_q) = 0 \quad \text{if } X \in \mathcal{H} \text{ } (\Omega \text{ is } \textit{horizontal}) \\ &\Omega([X,X_1], X_2, \ldots, X_q) + \ldots + \Omega(X_1, X_2, \ldots, [X, X_q]) = 0 \quad \forall\, X \in \mathcal{H} \quad . \end{aligned} \tag{7.2}$$

The cocycles and coboundaries are then defined by

$$Z^q(\mathcal{G},\mathcal{H}) = Z^q(\mathcal{G}) \cap C^q(\mathcal{G},\mathcal{H}) \quad , \quad B^q(\mathcal{G},\mathcal{H}) = sC^{q-1}(\mathcal{G},\mathcal{H}) \tag{7.3}$$

where s is the standard Lie algebra cohomology operator. The *relative Lie algebra cohomology groups* are now defined as usual,

$$H^q(\mathcal{G},\mathcal{H}) = Z^q(\mathcal{G},\mathcal{H})/B^q(\mathcal{G},\mathcal{H}) \quad . \tag{7.4}$$

Let us consider a horizontal LI form Ω on G and which is invariant under the right action of $\mathcal{H}$, namely

$$i_{X(g)}\Omega(g) = 0 \quad , \quad L_{X(g)}\Omega(g) = 0 \quad \forall\, X \in \mathcal{H} \tag{7.5}$$

Since there is a one-to-one correspondence between LI forms on Ω and multilinear mappings on $\mathcal{G}$, it is clear that (7.5) is the translation of (7.2) (Theorem 7.1) in terms of differential forms on the group manifold G.

Theorem 7.2
The ring of invariant forms on G/H is given by the exterior algebra of multilinear antisymmetric maps on $\mathcal{G}$ vanishing on $\mathcal{H}$ and which are $ad\,\mathcal{H}$-invariant.

Remark. Definition 7.1 requires to prove that $sC^q \subset C^{q+1}$. But this may be seen using that (7.2) may be written as $i_X\Omega(X_2,\ldots,X_q) = 0$ and $L_X\Omega(X_1,\ldots,X_q) = 0,\ X \in \mathcal{H}$. Now,

$$i_X(s\Omega)(X_1,\ldots,X_q) = (L_X - si_X)\Omega(X_1,\ldots,X_q) = 0 \tag{7.6}$$

and

$$L_X(s\Omega)(X_1,\ldots,X_q) = (sL_X)\Omega(X_1,\ldots,X_q) = 0 \tag{7.7}$$

since $s \sim d$, $si_X + i_Xs = L_X$ and $[L_X, s] = 0$.

Theorem 7.3
The Lie algebra cohomology groups $H^q(\mathcal{G},\mathcal{H})$ relative to $\mathcal{H}$ are given by the forms Ω on G which are a) LI b) closed and c) projectable.

Proof. LI means that they can be put in one-to-one correspondence with skew-linear forms on $\mathcal{G}$; closed implies that $d\Omega = 0$ or, in terms of the cohomology operator, that $s\Omega = 0$. Finally, projectable means that the relative cohomology conditions (7.2) are satisfied, *q.e.d.*

Note that, again, the relative and the de Rham cohomology on the coset may be different. However, if G is compact the following theorem [1, Theorem 22.1] holds

Theorem 7.4
Let G be a compact and connected Lie group, H a closed connected subgroup and K the homogeneous space $K = G/H$. Then $H^q(\mathcal{G},\mathcal{H})$ and $H^q_{DR}(K)$ are isomorphic, and so are their corresponding rings $H^*(\mathcal{G},\mathcal{H})$ and $H^*_{DR}(K)$.

The relative cohomology may be used to construct effective actions of WZW type on coset spaces [51, 52, 53]; the obstruction may be expressed in terms of an anomaly. For instance, when it is absent, the five cocycle on G/H has the form

$$\mathrm{Tr}(\mathcal{U}^5) - 5\mathrm{Tr}(\mathcal{W}\mathcal{U}^3) + 10\mathrm{Tr}(\mathcal{W}^2\mathcal{U}) \quad , \tag{7.8}$$

where $\mathcal{U}$ is the $(\mathcal{G}\backslash\mathcal{H})$-component of the canonical form θ on G and $\mathcal{W} = d\mathcal{V} + \mathcal{V}\wedge\mathcal{V}$ is the curvature of the $\mathcal{H}$-valued connection $\mathcal{V}$ given by the $\mathcal{H}$-component ω^α of θ. In fact, a similar procedure is also valid to recover the obstructions to the process of gauging WZW actions found in [55]. It may be seen that this is due to the relation between the relative Lie algebra cohomology and the equivariant (see [56]) cohomology, but we shall not develop this point here (see [57] and references therein).

Acknowledgements

This paper has been partially supported by a research grant from the MEC, Spain (PB96-0756). J. C. P. B. wishes to thank the Spanish MEC and the CSIC for an FPI grant. The authors also whish to thank J. Stasheff for helpful correspondence.

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